21. Answer: 7. Let N be a positive integer and d a divisor of N. Then $\frac{N}{d}$ is also a divisor of N. Thus the divisors of N occur in pairs $d, \frac{N}{d}$ and these two divisors will be distinct unless N is a perfect square and $d = \sqrt{N}$. It follows that N has an odd number of divisors if and only if N is a perfect square. There are 7 perfect squares among the numbers $1, 2, 3, \ldots, 50$.

    Note: If $N > 1$ is an integer then $N = {p_1}^{r_1} \cdot {p_2}^{r_2} \cdot \ldots \cdot {p_k}^{r_k}$ where p_i is the i^{th} prime. The divisors of N are those $d = {p_1}^{s_1} \cdot {p_2}^{s_2} \cdot \ldots \cdot {p_k}^{s_k}$ with $0 \le s_i \le r_i$ for all i. Thus, N has $(r_1+1) \cdot (r_2+1) \cdot \ldots \cdot (r_k+1)$ divisors, a product which will be an odd number only when each r_i is even.

22. **Answer (D):** Since $a \neq 0$, the only x for which $f(x) = -\sqrt{2}$ is $x = 0$. Since $f(f(\sqrt{2})) = -\sqrt{2}, f(\sqrt{2})$ must be 0. Thus $2a - \sqrt{2} = 0$, or $a = \frac{\sqrt{2}}{2}$.

    OR

    Since $f(\sqrt{2}) = 2a - \sqrt{2}, f(f(\sqrt{2})) = a(2a - \sqrt{2})^2 - \sqrt{2}$ which we set equal to $-\sqrt{2}$. Therefore, $a(2a - \sqrt{2})^2 = 0$. Since $a > 0, 2a = \sqrt{2}$ and $a = \frac{\sqrt{2}}{2}$.

23. Answer: 2880

    Each of the five rectagular side faces contributes 360° and the regular pentagon base and top each contribute $5 \cdot 108 = 540°$ for a total of $5 \cdot 360 + 2 \cdot 5 \cdot 108 = 2880$ degrees.

24. **Answer (A):** Angles BAC, BCD and CBD all intercept the same circular arc. Therefore $\angle BCD = \angle CBD = x$ and $\angle D = \pi - 2x$. The given condition now becomes $\frac{\pi - x}{2} = 2(\pi - 2x)$, which has the solution $x = \frac{3}{7}\pi$.

    OR

    Let O be the center of the circle. Then $\angle COB = 2x$ and, from the sum of the angles of the quadrilateral $COBD$, we obtain $2x + \angle D = \pi$. The conditions of the problem yield $x + 4\angle D = \pi$ to be the sum of the angles of $\triangle ABC$. Solve these two equations in x and $\angle D$ simultaneously to find $x = \frac{3\pi}{7}$.

    Query: What is x if $\triangle ABC$ is an obtuse isosceles triangle?

25. Answer: 89. Let the four numbers be w, x, y, and z with $w \le x \le y \le z$. Since each number appears three times in the four sums,

    $$3(w + x + y + z) = 180 + 197 + 208 + 222 = 807.$$

    Thus $w + x + y + z = 269$ and $w + x + y = 180$, so $z = 269 - 180 = 89$.

26. Answswer: 234. If all 26 people shook hands there would be $\binom{26}{2}$ handshakes. Of these, $\binom{13}{2}$ would take place between women and 13 between spouses. Therefore there were $\binom{26}{2} - \binom{13}{2} - 13 = 13 \cdot 25 - 13 \cdot 6 - 13 = 234$ handshakes.

27. Answer: 204. For every 3 distinct digits selected from $\{1, 2, \ldots, 9\}$ there is exactly one way to arrange them into a number with increasing digits, and every number with increasing digits corresponds to one of these selections. Similarly, the numbers with decreasing digits correspond to the subsets with 3 elements of the set of all 10 digits. Hence our answer is

$$\binom{9}{3} + \binom{10}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{9 \cdot 8}{2 \cdot 3}(7 + 10) = 12 \cdot 17 = 204.$$

OR

By making a list of such numbers with increasing digits in decreasing order,

$$789, \underbrace{689, 679, 678}, \underbrace{589, 579, 578, 569, 568, 567}, \ldots, \underbrace{189, \ldots, 124, 123}$$

and grouping them according to their first digit, we see there are $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + 7) = 84$ such numbers. Now make a list of such numbers with decreasing digits in increasing order:

$$210, \underbrace{310, 320, 321}, \underbrace{410, 420, 421, 430, 431, 432}, \ldots, \underbrace{910, \ldots, 986, 987}$$

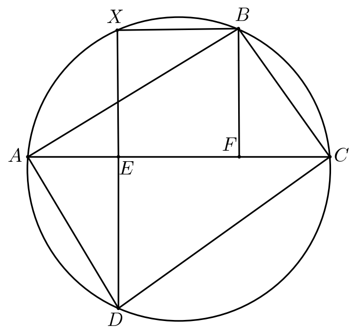
and group them according to their first digit. In this case there are $1 + 3 + 6 + 10 + \cdots + 28 + 36 = 120$ such numbers, an additional 36 numbers since a number with decreasing digits can contain 0 while one with increasing digits cannot. Thus, the answer is $84 + 120 = 204$.

28. **Answer (C):** By observing the repeating pattern of the units digits of consecutive integral powers of 3 we note that 25 of the given values for a yield a units digit in 3^n of 3, 25 yield 9, 25 yield 7 and 25 yield 1. Similarly, the given values for b yield 25 of each of these units digits in $7^b : 7, 9, 3, 1$. Thus there are 16 possible pairs of units digits, and each pair is equally likely. Of these, three pairs, $(1, 7), (7, 1)$ and $(9, 9)$, yield a sum with units digits 8. Thus, the desired probability is $\frac{3}{16}$.

29. Answer: 86. Since $\frac{N^2+7}{N+4} = \frac{(N-4)(N+4)+23}{N+4}$, the numerator and denominator will have a nontrivial common factor exactly when $N + 4$ and 23 have a factor in common. Because 23 is a prime, $N + 4$ is a multiple of 23 when $N = -4 + 23k$ for some integer k. Solving $1 < -4 + 23k < 1990$ yields $\frac{5}{23} < k < 86\frac{16}{23}$, or $k = 1, 2, \ldots, 86$.

30. **Answer (C):** Since $\angle BAF$ and $\angle ADE$ are both complementary to $\angle CAD$ they must be equal. Thus, $\triangle BAF \sim \triangle ADE$ so $\frac{BF}{AF} = \frac{AE}{DE}$, or $\frac{BF}{3+EF} = \frac{3}{5}$. By an analogous argument, $\triangle BCF \sim \triangle CDE$, $\frac{BF}{CF} = \frac{CE}{DE}$, and $\frac{BF}{7-EF} = \frac{7}{5}$. Solve these two equations simultaneously to obtain $BF = 4.2$.

OR



Note that $ABCD$ is a cyclic quadrilateral since opposite angles are supplementary. Extend $\overline{DE}$ to X on the circumcircle. Since $\angle DAB$ subtends the same arc as $\angle DXB$, $BFEX$ is a rectangle and $BF = EX$. We can consider $\overline{AC}$ and $\overline{DX}$ as intersecting chords in a circle and use $DE \cdot EX = AE \cdot EC$ to find $BF = EX = \frac{AE \cdot EC}{DE} = \frac{21}{5}$.